Business PreCalculus MATH 1643 Section 004, Spring 2014
Lesson 8: Other Types of Equations

Definition 1. **Solving Equations by Factoring:**

Step 1. *Make one side zero. Move all nonzero terms in the equation to one side so that the other side is zero.*

Step 2. *Factor the nonzero side.*

Step 3. *Use the Zero-Product Property. Set each factor in Step* 2 *equal to* 0 *and then solve the resulting equations.*

Step 4. *Check your solutions.*

Example 1. *Solve* $x^3 + 3x^2 = 4x + 12$.

$$\begin{aligned} x^3 + 3x^2 &= 4x + 12 && (\textit{original equation}) \\ x^3 + 3x^2 - 4x - 12 &= 0 && (\textit{Making one side equal to } 0) \\ x^2(x+3) - 4(x+3) &= 0 && (\textit{Group for factoring}) \\ (x+3)(x^2-4) &= 0 && (\textit{Factoring}) \\ (x+3)(x+2)(x-2) &= 0 && (\textit{Factor Completely}) \end{aligned}$$

Then by the Zero-Product *property, we have* $x + 3 = 0$, $x + 2 = 0$, *or* $x - 2 = 0$, *which implies that* $x = -3$, $x = -2$, *or* $x = 2$. *After checking the solutions in the original equation the solution set is* $\{-3, -2, 2\}$.

Definition 2. **Rational Equations:** *If at least one algebraic expression with the variable in the denominator appears in an equation then the equation is called a* rational equation. *For example, the equation* $\frac{1}{6} + \frac{1}{x+6} = \frac{1}{x}$ *is a rational equation.*

Definition 3. **Extraneous Solution:** *When we multiply a rational equation by an expression containing the variable, we may introduce a solution that satisfies the new equation but* does not *satisfy the original equation. Such a solution is called an* **extraneous solution** *or* **extraneous root**.

Definition 4. **Solving Rational Equations:**

Step 1. *Find the LCD. The LCD from the denominator.*

Step 2. *Multiply both sides of the equation by LCD and make one side equal to* 0.

Step 3. *Factor to solve (or use any other method).*

Step 4. *Set each factor equal to* 0 *(in case you factored in Step 3)*

Step 5. *Check the solutions in the original equation. (VERY IMPORTANT TO DECIDE WHETHER THERE ARE EXTRANEOUS SOLUTIONS OR NOT).*

Example 2. *Solve the following rational equation:* $\frac{x}{x-1} - \frac{1}{x+1} = \frac{2x}{x^2-1}$.
Solution: **Step 1.** *The LCD of the denominators* $(x-1), (x+1), (x^2-1) = (x-1)(x+1)$ *is* $(x-1)(x+1)$.

$$(x-1)(x+1)[\frac{x}{x-1} - \frac{1}{x+1}] = (x-1)(x+1)[\frac{2x}{x^2-1}] \text{ (Multiply both sides by the LCD)}$$

$$(x-1)(x+1)\frac{x}{x-1} - (x-1)(x+1)\frac{1}{x+1} = (x^2-1)\frac{2x}{x^2-1}$$

$$(x+1)x - (x-1) = 2x \text{ Simplifying}$$

$$x^2 + x - x + 1 = 2x$$

$$x^2 + 1 = 2x$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)(x-1) = 0$$

Hence, $x - 1 = 0$ *or* $x = 1$. *Finally, substituting* $x = 1$ *in the original equation* $\frac{x}{x-1} - \frac{1}{x+1} = \frac{2x}{x^2-1}$ *yields* $\frac{1}{1-1} - \frac{1}{1+1} = \frac{2(1)}{1^2-1}$ *or* $\frac{1}{0} - \frac{1}{2} = \frac{2}{0}$. *Therefore,* $x = 1$ *is an extraneous solution.*

Definition 5. **Solving Equations Involving a Radical:**

Step 1. *We begin by isolating the radical to one side of the equation.*

Step 2. *Next, we square both sides of the equation and simplify.*

Step 3. *Set each factor equal to zero.*

Step 4. *Check the answers in the original equation.*

Example 3. *Solve:* $\sqrt{2x+1} + 1 = x$.
Solution:

$$\sqrt{2x+1} + 1 = x$$

$$\sqrt{2x+1} = x - 1 \qquad \text{Isolating the radical}$$

$$(\sqrt{2x+1})^2 = (x-1)^2 \qquad \text{Squaring both sides}$$

$$2x + 1 = x^2 - 2x + 1$$

$$x^2 - 4x = 0 \qquad \text{Simplifying}$$

$$x(x-4) = 0 \qquad \text{Factoring}$$

Hence, $x = 0$ *or* $x = 4$. *Finally, plugging* $x = 0$ *in* $\sqrt{2x+1} + 1 = x$ *yields* $\sqrt{2(0)+1} + 1 = 0$ *or* $2 = 0$ *which is impossible, while plugging* $x = 4$ *in* $\sqrt{2x+1} + 1 = x$ *yields* $\sqrt{2(4)+1} + 1 = 4$ *or* $4 = 4$. *Therefore, the set of solutions of the equation* $\sqrt{2x+1} + 1 = x$ *is* $\{4\}$.

Definition 6. **Solving Equations with Rational Exponents:** *If a given equation can be expressed in the form* $u^{\frac{m}{n}} = k$, *where* m *and* n *are positive integers. Then* $[u = k^{\frac{n}{m}}$: *If* m *is odd*$]$ *and* $[u = \pm k^{\frac{n}{m}}$: *If* m *is even*$]$ *i.e raising to the power* $\frac{n}{m}$ *the reciprocal of* $\frac{m}{n}$.

Example 4. *Solve:* $2(2x-1)^{\frac{3}{2}} - 30 = 24$.
Solution:

$$\begin{aligned}
2(2x-1)^{\frac{3}{2}} - 30 &= 24 \\
2(2x-1)^{\frac{3}{2}} &= 24 + 30 = 54 \\
(2x-1)^{\frac{3}{2}} &= \frac{54}{2} = 27 \\
2x - 1 &= 27^{\frac{2}{3}} \\
2x - 1 &= (3^3)^{\frac{2}{3}} = 3^2 = 9 \\
2x &= 9 + 1 = 10 \\
x &= \frac{10}{2} = 5
\end{aligned}$$

Definition 7. Equations that are Quadratic in Form: *An equation in a variable x is* quadratic in form *if it can be written as*

$$au^2 + bu + c = 0 \qquad (a \neq 0)$$

where u is an expression in the variable x.

Example 5. *Solve:* $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$.
Solution: *The equation is not a quadratic equation. However, if we let $u = x^{\frac{1}{3}}$, then $u^2 = (x^{\frac{1}{3}})^2 = x^{\frac{2}{3}}$. The original equation becomes a quadratic equation if we replace $x^{\frac{1}{3}}$ by u.*

$$\begin{aligned}
x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 &= 0 \qquad \textit{original equation} \\
u^2 - 5u + 6 &= 0 \qquad \textit{Replace } x^{\frac{1}{3}} \textit{ with } u \\
(u-2)(u-3) &= 0
\end{aligned}$$

Then, $u = 2$ or $u = 3$ which implies that $x^{\frac{1}{3}} = 2$ or $x^{\frac{1}{3}} = 3$. Hence, $x = 2^3 = 8$ or $x = 3^3 = 27$.